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Priorisierung in der Medizin
FOR 655 Nr. 05 / 2008



Die Reihe „Priorisierung in der Medizin“ umfasst Arbeits- und Forschungsberichte der DFG Forschergruppe FOR655 „*Priorisierung in der Medizin: eine theoretische und empirische Analyse unter besonderer Berücksichtigung der Gesetzlichen Krankenversicherung (GKV)*“.

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Impressum:



Campus Ring 1
28759 Bremen
Germany
www.jacobs-university.de

ISSN 1866-0290

www.for655.de | www.priorisierung-in-der-medizin.de

A Cooperative Bargaining Model for two Groups of Patients

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We discuss an allocation problem for a perfectly divisible good that is to be allocated to two groups of patients. Each group has an individual need of that good. If that group receives the requested amount it gains a higher quality of life than if not receiving the amount. The total amount available to share is not enough to serve all patients in all groups. We consider this allocation problem in the context of cooperative bargaining theory and show that for two representative patients we have a unique solution that satisfies the axioms symmetry, weak Pareto optimality and invariance under positive affine transformation. We show that the solution is equal to the Nash and the Kalai-Smorodinsky solution. The allocation mechanism assigns equal chances to receive the requested amounts.

Introduction

The model in this paper refers to a medical allocation problem that has been discussed in philosophic literature since the late 70ies of the 20th century. In detail, the Parfit-Taurek discussion (e.g. in Lübbe (2004)) is about Taurek's example of saving the life of 5 individuals versus the life of 1 with the same amount of medicine. This illustrates a dilemma situation in allocating medical resources to patients. Ahlert (2006) generalizes Taurek's example in health economics and defines an allocation problem with a finite number of patients who each have an individual need. The total amount of the medical good does not suffice to fulfill all needs. This paper considers this type of models and defines and analyzes solutions to such allocation situations. We apply cooperative bargaining theory and characterize allocation mechanisms by their normative properties.

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Finding solutions to the described allocation problem one may consider different solution concepts. First, we may apply a stochastic process that chooses patients with a certain probability. Ahlert (2006) and Ahlert (2007) define such a random mechanism for the allocation problem. A second approach considers allocating medical resources according to a priority list as proposed by Ahlert (2005) and Zimmermann (2007). Finally, we contribute to the discussion by applying a cooperative bargaining approach to the allocation problem. That is, we model the allocation problem as a bargaining problem and we claim normative axioms for a solution that are derived from cooperative bargaining theory. In particular, we focus on the bargaining concepts of Nash (1950) and Kalai-Smorodinsky (1975).

The paper is organized as follows. In section 2 we define assumptions and notations of the model such as a chance of receiving a medical good and success probabilities under the condition of treatment or no treatment. In section 3 we introduce a cooperative bargaining problem for two groups with one patient in each group. We model a set of chances of success which we denote the bargaining set and define a status quo in bargaining, a bargaining problem, and solution. Section 4 discusses the axioms of symmetry, weak Pareto optimality and invariance under positive affine transformation. We show that there exists a unique solution that satisfies the three axioms. Furthermore, we prove that the solution is equal to the Nash and the Kalai-Smorodinsky solution. Finally, we summarize the results of the paper and give implications for further research.

1. Definition of an allocation problem

The situation in the model represents a situation in health economics. One can think of the process of allocating medical resources to departments in a hospital or to resorts of national health care services.

Patients are modelled in patients groups which are understood as their assigned diagnosis-treatment-pairs. Patients in one group have similar diagnosis and treatment related characteristics such as illness, treatment catalogue and success probabilities in case of being treated or in case of not being treated. As a result, we assume that patients in one group are homogenous in these characteristics and heterogeneous compared to other groups. Each group may be headed by a representative that acts on behalf of the patients in that group. This person has perfect knowledge of the type of illness and possible treatments for patients in that group. For example, dialysis patients are treated in one department of a hospital. These patients' interests (medical needs) are represented by the head of department which is typically an experienced doctor. By assumption this person knows the medically indicated needs of dialysis patients treated in his department and the expected number of patients per time unit considered. He claims a portion of the budget equal to sum of these needs.

For this paper we introduce a model for two patients groups i with $i = 1, 2$ and each group has one representative patient. Of course, the model can be generalized to more than two diagnosis-treatment pairs and to more than one patient in each group. As we

assume to have only one patient in each group we may simplify notations. When we use the notation of group i we refer to the representative patient of that group.

We denote by q the amount of money that pays for a certain supply of medical goods and services. The amount of medical supplies q is considered perfectly divisible. For example, a hospital is assigned a fixed amount of money annually, which restricts the number of medical services of the hospital in that year. The amount of medical resources can be interpreted in monetary units. However, we use the term medical resources and medical treatment in relation to the medical allocation problem and keep in mind that medical resources, of course, are financed by a monetary budget.

We denote by q_i the amount of medicine of the representative person of group i . For the representative patient in each group i we are given a medical need q_i towards the medical resource. The amount each patient requests is determined by an expert not by the patient herself. The need is such that a patient must receive this amount. Giving more or less is a waste of resource and considered a futile allocation as defined by Ahlert (2006).

The allocation problem is such that the total amount of medical supply available is not enough to treat the representative patients of both groups. Therefore, the restriction is such that $\sum_{i=1,2} q_i > q$.

For the patient in each group i we are given success probabilities. We call the success probability for a representative patient i with treatment s_i and the success probability without treatment r_i . A probability of success without treatment characterizes the health state and the quality of life of that person for a given period of time, when the patient does not receive the requested good. A probability of success with treatment, on the other hand characterizes the health state and the quality of life of that person for the given period, if the patient receives treatment. We simplify the complexity of the increase of health by assuming that health status can be represented by one dimensional parameter, the probability to become “healthy”. We assume that receiving treatment significantly increases the health state and the quality of life of the person for some time compared to a situation when no treatment is given. There are situations, e.g. in palliative care, when one cannot increase patients’ health state significantly yet this patient is still treated. Nonetheless we focus in this paper on situations in which we can significantly increase patient’s health levels. We assume that any medical treatment will increase the quality of life for that individual such that $0 \leq r_i < s_i \leq 1$. Success probabilities are determined by an experienced doctor, for example, who is able to estimate average quality of life for a patient treated in his department when this patient receives or does not receive treatment.

For two groups with one representative patient each a vector $s = (s_1, s_2)$ represents the probabilities of success with treatment and $r = (r_1, r_2)$ represents the probabilities of success without treatment such that $r < s$, componentwise, i.e. $r_1 < s_1$ and $r_2 < s_2$.

We define an allocation problem a for two groups i with one patient in each group, the individual needs of these patients q_i and their success probabilities with and without treatment s_i and r_i according to Ahlert (2006) such that $a = (q; q_1; q_2; r_1; r_2; s_1; s_2)$. The set of all allocation problems for two groups is denoted by A_2 .

For the allocation problem a we determine feasible deterministic allocations for a given supply of medical goods and individual needs. A feasible deterministic allocation is a vector $x = (x_1, x_2)$ such that $x_i \in \{0,1\}$ and $\sum_{i=1,2} x_i q_i \leq q$. Since we do not consider futile allocations, only two situations become available. First, $x_i = 1$ what means that patient i receives its requested amount and patient $j \neq i$ does not, i.e. $x_j = 0$. Second, $x_i = 0$ that means patient i receives nothing and patient $j \neq i$ receives the needed amount, i.e. $x_j = 1$. Each feasible deterministic allocation x represents an allocation where one patient receives the requested amount of the medical good and the other does not. In addition, of course, the allocation where nobody is treated is feasible, too. The set of all feasible deterministic allocations x of an allocation problem $a \in A_2$ is denoted by F . By allowing lotteries on the set of deterministic feasible allocations we will enlarge the set of feasible allocations such that we will construct a feasible set of allocations that is the convex closure of F .

An example

Consider the situation $q = 10$, $n = 2$, $q_1 = 3$, $q_2 = 9$ This is an example of two groups with only one patient in each group. Feasible deterministic allocations of the example are $(0,0)$, $(1,0)$, and $(0,1)$. Efficient allocations are $(1,0)$ and $(0,1)$ such that only one of the two patients receives the requested amount.

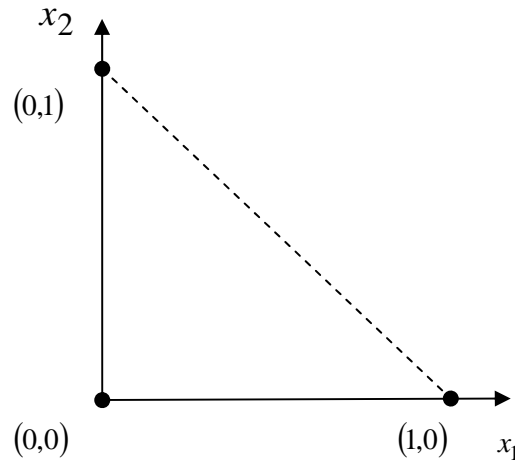


Figure 1: **Feasible deterministic allocations with** $q = 10$, $n = 2$, $q_1 = 3$, $q_2 = 9$

The given allocation problem a generates deterministic allocations such that each allocation $x \in F$ could help one of two patients. In allocation situations in which we cannot help all, we may demand that each need is at least considered in the allocation process. Each patient with a specified need that could in principle be satisfied ought to have a chance to receive the medical good under the allocation mechanism.

Ahlert (2006) defines a random allocation mechanism over feasible deterministic allocations. This generates chances of receiving the requested amount. A random allocation for a given allocation problem $a \in A_2$ assigns a probability distribution on the set of feasible deterministic allocations F and a random allocation rule h applied to a chooses a probability distribution $h(a) = p$ on F . The set P contains of all probability distributions p on F .

We consider the set of feasible deterministic allocations F of an allocation problem a and a probability distribution p on F which chooses a feasible deterministic allocation $x \in F$ with $p(x)$.

For a given probability distribution p on F we denote by $c_i(p)$ the chance that patient i receives the good. The chance that she receives the good is the sum of probabilities over all chances of feasible allocations with $x_i = 1$ such that $c_i(p) = \sum_{x \in F} x_i p(x)$. For the chance to receive the good $0 \leq c_i(p) \leq 1$ holds.

The vector $c = (c_1(p), c_2(p))$ for a given P then represents a feasible chance allocation for two patients. C denotes the set of all feasible chance allocations. Each vector $c \in C$ is originated by some probability distribution p on F . The set C is generated by the set of all lotteries over feasible deterministic allocations and is represented by the convex closure of F (Ahlert 2006).

For the introduced example from Ahlert (2006), the chance allocation problem is shown in Figure 2.

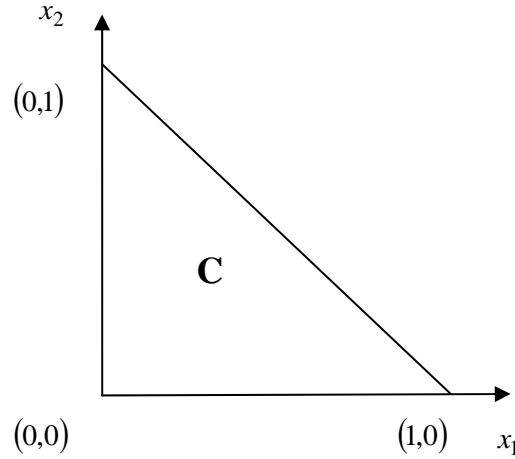


Figure 2: A set of feasible chance allocations

In addition to a chance of receiving the good $c_i(p)$, we denote the chance of not receiving the good by $\bar{c}_i(p)$. Then a chance of not receiving the good is $\bar{c}_i(p) = 1 - c_i(p)$.

Before we define chances of success, we need to refine the definition of probabilities of success with and without treatment. We understand a probability of success with treatment s_i to be a chance of success (S) under the condition of being treated (T).

Definition 1: A conditional chance of success s_i if patient i is treated is defined by $s_i = P(S|T) = P(S \cap T) / P(T) = s_i \cdot c_i(p) / c_i(p)$.

We understand a probability of success without treatment r_i to be a chance of success (S) under the condition on not being treated (\bar{T}).

Definition 2: A conditional chance of success r_i if patient i is not treated is defined by $r_i = P(S/\bar{T}) = P(S \cap \bar{T})/P(\bar{T}) = r_i \cdot (1 - c_i(p))/(1 - c_i(p))$.

According to our definition, a chance of success for the patient in group i is the combined probability of chances of success with and without treatment. The probability of success l_i is such that

$$l_i = P(S) = P(S \cap \bar{T}) + P(S \cap T) = s_i \cdot c_i(p) + r_i \cdot (1 - c_i(p)).$$

Rearranging the above equation, we derive $l_i = c_i(p) \cdot (s_i - r_i) + r_i$. Under the assumption that probabilities r_i and s_i are given, a chance of success is a positive affine transformation of chances of receiving treatment. The term $(s_i - r_i)$ represents the gain in chances of success when a patient receives treatment. By definition it holds that $0 < (s_i - r_i) < 1$. A group receives a positive increase in chances of success when she is allocated. A one percent increase in chances of receiving treatment results in an increase in chances of success by $(s_i - r_i)$ percent.

A chance of success for the patient in group i determines an expected living quality as indicated by her health state. The expected health state of that patient depends on his success probabilities and whether or not she is treated. A patient's success probability indicates the chances to recover from an illness in a given period of time.

2. A bargaining problem of chances of success

We define a set of allocations of chances of success for two groups with one patient in each group. We start with the deterministic allocations: each feasible deterministic allocation $x \in F$ can be represented by a chance vector of receiving the good such that the components $c_i(p) \in \{0,1\}$ for at most one $i = 1,2$. We can further determine a chance of success $l_i = c_i(p) \cdot (s_i - r_i) + r_i$ for such values of $c_i(p)$. For each chance of receiving the good $c_i(p) = 1$ we generate a chance of success $l_i = s_i$ and for $c_i(p) = 0$ we receive $l_i = r_i$. We call y a feasible chance allocation of success with $y = (y_1, y_2)$ and $y_i \in \{r_i, s_i\}$ with $i = 1,2$. For each $x \in F$ we derive a chance vector y . The set of all chance vectors y generated from deterministic feasible allocations is called G . For the given allocation problem, feasible allocations of chances of success in G are $\{(r_1, r_2); (r_1, s_2); (s_1, r_2)\}$.

We consider a set of feasible chance allocations of success G of an allocation problem a and a probability distribution q on G which chooses a chance allocation of success $y \in G$ with $q(y)$. The set Q contains all probability distributions q on G .

A vector $l = (l_1(q), l_2(q))$ for a given distribution q on G represents a feasible chance allocation of success for two patients. A set L is the set of all feasible chance allocations for chances of success. Each vector $l \in L$ is originated by a probability distribution q on G . A set L is determined by a lottery over feasible deterministic allocations y with a probability distribution q on G . And L is a convex closure of G like C was the convex closure of F (Ahlert (2006)). We understand a set L to be an alternative set in a cooperative bargaining model.

The chance of success for patients in group i is the sum of probabilities over all feasible allocations such that $l_i(q) = \sum_{y \in G} y_i q(y)$. The chance of success $l_i(q)$ for group i is such that $0 \leq r_i \leq l_i \leq s_i \leq 1$ under the restriction that $r_i < s_i$.

When no patient receives treatment, i.e. $c_i(p) = 0$, all realize an expected health state $l_i = r_i$ for $i = 1, 2$. That is the situation when no representative is treated is denoted by r . Given a set L , we call this allocation the status quo $r = (r_1, r_2)$. From the definition of L we know that $r \in L$.

When both groups are treated with certainty, we arrive at $l = s$. We understand the vector $s = (s_1, s_2)$ as the vector of best possible health states that all patients can attain in case they all would receive treatment. From the definition of feasible chances of success for an allocation problem a we conclude that s is not a feasible chance vector in the set of feasible chances of success L such that $s \notin L$. This is obvious because treating all individuals of all groups is impossible under the definition of the allocation problem a . We call this vector s the ideal point.

Before we discuss feasible sets of chances of success under the given allocation problem, we want to clearly exclude the situation where both patients can be treated. This set would be the convex hull of the allocations of chances of success $\{(r_1, r_2); (r_1, s_2); (s_1, r_2); (s_1, s_2)\}$. Since we have $s \notin L$ such a set is not a feasible set of chances in this model.

Dependent on the type of the allocation problem a we may have different types of sets of possible chance of success L .

First, a feasible set is for example derived by a situation when no patient receives treatment. Then we arrive at $L = \{(r_1, r_2)\}$. Second, we receive two more sets of chances of success when the situation is such that we can help either one patient or the other. In case we can only treat the patient in group 1 because the quantity patient 2 needs exceeds the amount available, then we have a set of chances of success that is the convex hull of the following chances allocations such that $L = ch\{(r_1, r_2); (s_1, r_2)\}$. On the other hand, it is possible that only the patient in group 2 can receive treatment. Then the set of chances of success is such that $L = ch\{(r_1, r_2); (r_1, s_2)\}$.

Figure 3 shows a feasible set of chances of success $L = ch\{(r_1, r_2); (r_1, s_2); (s_1, r_2)\}$. We receive such a set when all representative patients have the chance to improve their health status, i.e. under the given allocation problem it is feasible to treat them.

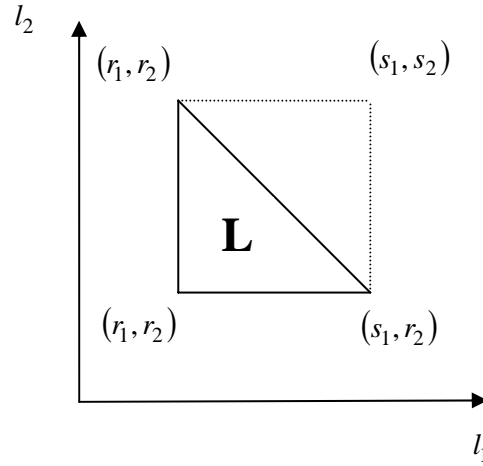


Figure 3: A set of feasible chances of success

For the allocation situation we assume that there exists an $l \in L$ such that $l > r$ (componentwise such that $l_i > r_i$ for $i = 1, 2$), i.e. each representative has an individual rational incentive to bargain. If we admit only patients that have a chance to improve their expected health status compared to the status quo, then representative patients have an incentive to agree to the application of this allocation mechanism. As a result, representatives whose needs are greater than the total amount of medical resource cannot be treated and are excluded from the allocation as well as representatives with no significant increases in success probabilities with treatment.

Due to the definition of individual rationality, we restrict feasible sets of chances of success to sets $L = ch\{(r_1, r_2); (r_1, s_2); (s_1, r_2)\}$ as illustrated in Figure 3. Since we have defined a chance of success l_i for $i = 1, 2$ in the interval $[0, 1]$, each set L is a triangle situation and such sets form a restricted two-dimensional space which we denote by \tilde{R}^2 .

Definition 3: A pair (L, r) with $L \in \tilde{R}^2$ and $r \in L$ is called a bargaining problem, if there is at least one $l \in L$ such that $l > r$.

We denote the class of all bargaining problems for two representative patients constructed as described above by \tilde{B}^2 . The class of bargaining problems \tilde{B}^2 considered in this paper contains all triangular problems (L, r) with chance set $L \in \tilde{R}^2$ and $r \in L$.¹

Definition 4: A function $f : \tilde{B}^2 \rightarrow \tilde{R}^2$ such that for every $(L, r) \in \tilde{B}^2$ $f(L, r) \in L$ holds is a solution on \tilde{B}^2 .

We interpret the solution of the bargaining problem as an allocation of expected chances of success that represents an expected health state for each representative patient. For a given allocation problem the solution to the bargaining problem is a consequence of a social agreement on desired properties such that the allocation mechanism determines an expected health state for each group.

3. Axiomatic characterization and solution concepts

We model desired properties of an allocation rule in the framework of cooperative bargaining theory and define solution concepts that fulfil these properties.

Since \tilde{B}^2 consists of different triangular situations with status quo at the origin of the rectangle we have to model how the allocation should react on changes in the status quo or in the gain of success of each patient. We will assume that the representatives do not make interpersonal comparisons of the probabilities. That means if e.g. r_1 and s_1 change by some positive affine transformation this should not effect the allocation to patient 2 and vice versa. This implies that the allocation to patient 1 is changed according to his or her transformation. Allowing positive affine transformations for both patients leads to the axiom of invariance known from bargaining without interpersonal comparisons of utilities.

We consider a bargaining problem (L, r) from \tilde{B}^2 with success probabilities r_i and s_i for each patient and assume that these probabilities shift to \tilde{r}_i and \tilde{s}_i for $i = 1, 2$ such that we receive a vector of individual shifts for all patients \tilde{r} and \tilde{s} . Then the corresponding chances of success l_i shift to \tilde{l}_i for $i = 1, 2$. For each allocation of chances of success $l \in L$ we receive a chance allocation \tilde{l} . A set of chances of success for the new situation is called \tilde{L} such that all $\tilde{l} \in \tilde{L}$.

¹ The class \tilde{B}^2 considered in this paper is a restricted domain of bargaining problems. Due to restrictions of chances of success l_i the class \tilde{B}^2 contains fewer problems than the unrestricted class of bargaining problems B^2 for two persons.

Axiom INV (Invariance under positive affine transformation of success probabilities):

Given a bargaining problem $(L, r) \in \tilde{B}^2$ and new probabilities of success \tilde{r}_i and \tilde{s}_i with $0 \leq \tilde{r}_i < \tilde{s}_i \leq 1$ let the bargaining problem (\tilde{L}, \tilde{r}) be defined by $\tilde{L} = \left\{ \tilde{l} \in \tilde{R}^2 \mid \exists l \in L \text{ such that } \tilde{l}_i = (\tilde{s}_i - \tilde{r}_i) \cdot l_i + \tilde{r}_i \text{ for } i = 1, 2 \right\}$.

Then $f_i(\tilde{L}, \tilde{r}) = (\tilde{s}_i - \tilde{r}_i) \cdot f_i(L, r) + \tilde{r}_i$ for $i = 1, 2$.

A solution to the bargaining problem shifts analogously to the vector of individual shifts of probabilities of success of the representative patients. This axiom in the last consequence implies that every situation is equivalent to the triangle $\text{ch}\{(0,0), (0,1), (1,0)\}$. That means that the two representatives bargain about chances of treatment independent of the values of r_i and s_i .

The next axiom is a well known and socially desirable requirement for allocations.

Axiom WPO (Weak Pareto optimality):

Given any $(L, r) \in \tilde{B}^2$. If $l \in L$ and there exists another chance allocation $k \in L$ such that $k > l$ holds (componentwise: $k_i > l_i$ for $i = 1, 2$), then $l \neq f(L, r)$.

Weak Pareto optimality is desirable because it requires a solution to the bargaining problem such that one cannot achieve better quality of life for both groups simultaneously. It may be possible to improve the living quality for one group with the other group remaining at the same level of living quality. Axiom WPO implies that a solution to the bargaining problem is situated on the weak Pareto optimal border of the chance set (which is in the cases considered here identical to the set of strongly Pareto optimal allocations) which is visualized as a bold line in Figure 4.

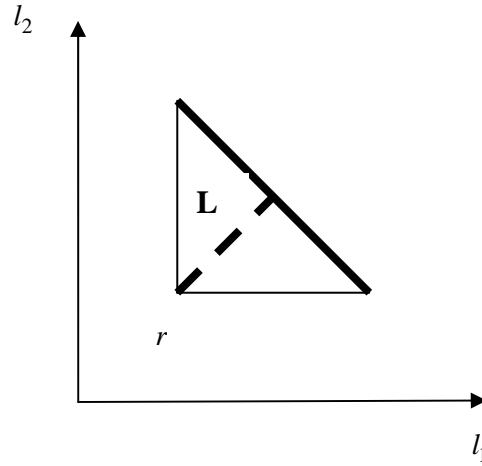


Figure 4: Weak (strong) Pareto efficient set and symmetric alternatives

Axiom SYM (Symmetry):

Let (L, r) be a symmetric bargaining problem in \tilde{B}^2 ($r_1 = r_2$ and $s_1 = s_2$), i.e. for each $l \in L$ every permutation of l is also in L . Then $f_1(L, r) = f_2(L, r)$.

If we have a symmetric bargaining problem, we cannot distinguish between representatives with respect to their success probabilities and chances to receive treatment. Yet groups may differ in the type of illness, however, this information is not included in the representation of the bargaining problem. Then the axiom requires that the solution to the problem is also symmetric. In our model a symmetric solution assigns equal expected increase of health states compared to the status quo to both groups. A solution to a symmetric bargaining problem must be an allocation on the dashed line in Figure 4.

In the following we give a definition of two solution concepts from cooperative bargaining of two-person problems on the unrestricted domain of bargaining problems B^2 . First, we introduce the Nash solution which is uniquely defined by the axioms of invariance under positive affine transformations, symmetry, weak Pareto optimality and independence of irrelevant alternatives (IRR). We defined the first three for bargaining problems on \tilde{B}^2 and they are reductions of the general axioms and it is known from literature (e.g. Thomson (1994)) that these axioms apply to B^2 . The Axiom of independence of irrelevant alternatives states that a solution to a bargaining problem does not change when certain feasible alternative are no longer available under a new problem. This holds only under the condition that the status quo in both situations is

identical and the solutions are feasible alternatives in the old and the new situation. As we study triangular situations this axiom becomes obsolete.

Theorem 1: (Nash (1950)) The Nash solution is the only solution on B^2 satisfying INV, WPO, SYM, and IRR.

Proof. We omit the proof. It may be studied in Thomson (1994).

Second, we define the Kalai-Smorodinsky solution which is characterized on B^2 by the axioms of invariance under positive affine transformations, weak Pareto optimality, symmetry and restricted monotonicity (RMON). Again, the first three axioms are reductions of general axioms on B^2 (Roth(1979a)). The axiom states that a solution in two different bargaining situations that do not differ in their ideal points and status quos a solution to the larger bargaining problem is at least as big as in the smaller problem. Since we study triangular situations this axiom becomes obsolete.

Theorem 2: (Kalai-Smorodinsky (1975)) The Kalai-Smorodinsky is the only solution on B^2 satisfying INV, WPO, SYM, and RMON.

Proof. We omit the proof. It may be studied in Roth (1979a).

Both solution concepts fulfil the axioms of INV, WPO, and SYM on B^2 . For any triangular situation on B^2 these three axioms define a unique solution on B^2 . We use INV to shift the original problem to a symmetric bargaining problem. Applying SYM and WPO to this problem leads to a unique solution. The only solution with equal coordinates that is weakly Pareto optimal must be the midpoint of the hypotenuses of the triangular situation. The results of the general class hold on any subclass of B^2 . The restricted domain of bargaining problems \tilde{B}^2 considered in this paper is a subclass of bargaining problems in B^2 such that $\tilde{B}^2 \subseteq B^2$. Then for any bargaining problem (L, r) in \tilde{B}^2 , we receive an analogous result.

Theorem 3: There is a unique bargaining solution on the class of bargaining problems \tilde{B}^2 that satisfies the axioms of INV, WPO, and SYM. This solution is equivalent to the Nash and the Kalai-Smorodinsky solution on \tilde{B}^2 .

The proof proceeds in three steps. First, we show that the required properties imply the uniqueness of the solution on \tilde{B}^2 . Second, we show that this solution is equal to the Nash and Kalai-Smorodinsky solution. Third, we show that the unique solution fulfils the three axioms.

Proof. We consider any bargaining problem (L, r) from the class of problems \tilde{B}^2 . By INV we may construct a bargaining problem (L', r') such that $r' = (0,0)$ and

$L' = ch\{(0,0);(0,1);(1,0)\}$. The bargaining problem is symmetric. A solution to a symmetric bargaining problem is an allocation in L' with equal coordinates for $i = 1,2$. We construct a set of symmetric allocations L^{SYM} such that $L^{SYM} = ch\{(0,0);(1,1)\}$. We denote the set of weakly Pareto optimal allocations in L' with L^{WPO} such that $L^{WPO} = ch\{(0,1);(1,0)\}$. The intersection of the weakly Pareto optimal set and the symmetric set $L^{SYM} \cap L^{WPO}$ gives us allocations of equal coordinates that are weakly Pareto optimal. The only weakly Pareto optimal allocation with equal coordinates is determined by a lottery on L^{WPO} which assigns each deterministic allocation a probability of $1/2$. Therefore, $L^{SYM} \cap L^{WPO} = \{(1/2,1/2)\}$. It follows that $f(L',0) = (1/2,1/2)$. By INV we may transform the solution of the situation $(L',0)$ into any other solution of a bargaining problem (L,r) within \tilde{B}^2 . This shows that the unique solution always chooses the midpoint of the hypotenuse for situations in \tilde{B}^2 . This completes the first part of the proof.

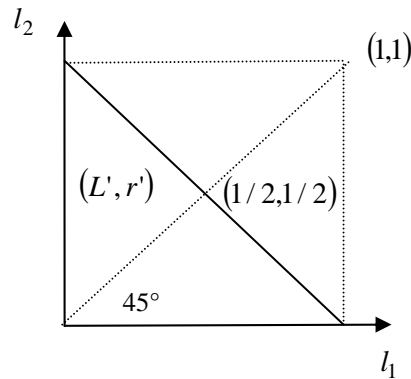


Figure 5: Bargaining problem (L', r') for $n=2$.

Next, we show that the unique solution is equal to the Nash and Kalai-Smorodinsky solution restricted to \tilde{B}^2 . It is known from the literature and easy to see from the properties of the Nash and Kalai-Smorodinsky solution that applied to triangle situations with status quo in the rectangle they both pick the midpoint of the hypotenuse as the solution. This means that any solution on \tilde{B}^2 fulfilling the three axioms is identical to the Nash and to the Kalai-Smorodinsky solution.

The third part is to recall that the Nash and the Kalai-Smorodinsky solution fulfil the axioms of symmetry and weak Pareto optimality on the set B^2 of all bargaining situations with two parties and therefore especially fulfil these properties on \tilde{B}^2 . The

axiom of invariance under positive affine transformations in our context allows less positive affine transformations than the general axiom for bargaining solutions on B^2 since translations are restricted not to exceed the interval $[0,1]$. Since the Nash and the Kalai-Smorodinsky solution both fulfill the stronger requirement on B^2 they also fulfill the weaker requirement on the smaller domain \tilde{B}^2 . Thus the solution function f on \tilde{B}^2 fulfils the axioms of INV, SYM, and WPO. This completes the proof.

4. Results and Conclusion

We apply bargaining theory to the simple case of two groups with one patient in each group. By assumption these two patients are representatives for patients in their groups. Here, the bargaining concepts of Nash and Kalai-Smorodinsky lead to the same unique solution. Each patient receives treatment with the probability of $\frac{1}{2}$ such that a patient realizes equal (proportional) expected gains in living quality compared to the status quo. This is equivalent to equal (proportional) expected decreases in living quality compared to the ideal point. In this special case the solution concepts of Nash and Kalai-Smorodinsky represent a special egalitarian principle.

On the set of the considered two-person cases the above solution mechanism can be uniquely characterized by the properties of weak Pareto efficiency, symmetry, and invariance under positive affine transformations. In the special case of bargaining situations considered here the concepts of weak and strong Pareto optimality lead to the same allocations. According to strong Pareto efficiency, expected health states of the solution allocation can not be improved individually without decreasing another patient's health state. The property of symmetry requires that identical conditions in the allocation problem lead to identical allocations of expected living quality under the solution concept. Transformation invariance refers to applicable positive affine transformations of success probabilities of each patient individually. The stochastic solution is independent of such positive affine transformations of individual success probabilities.

Further research on situations with two patient groups with different group sizes is planned. This must be done in another model which allows different numbers of basically homogenous types of individuals in each group together with their needs. Then we can again analyze the situation by applying cooperative bargaining theory. It is obvious that on this space the concepts of Nash and Kalai-Smorodinsky lead to different solutions. It is one of our research questions how both solutions can be characterized by normative properties in this space.

A further step may be the extension to more than two groups of patients. The generalization of the Nash solution will probably be straight forward. However, the property of strong Pareto efficiency is problematic for the concept of Kalai-Smorodinsky with more than two groups as it is with more than two persons (Roth (1979a)). We will probably have to apply a lexicographic solution concept (c.f. Imai (1983)). This implies that first the needs of all patients in all groups are satisfied according to some egalitarian principle that still needs to be derived. Afterwards the expected health states of some patients can be increased further on in order to achieve Pareto optimality.

***Author note**

This paper presents results of a research project on priorities, equality of chances, justice and efficiency in allocating medical resources. This project TP C04 is one of the projects on health care in Group FOR 655 (an interdisciplinary group of researchers) which are financed by DFG (German Research Foundation). The financial support by DFG is gratefully acknowledged.

The author would like to thank Prof. Dr. Dr. Marlies Ahlert, Prof. Dr. Bettina Klaus and Jana Zimmermann for helpful comments and review.

References

Ahlert, Marlies (2005): Priorities in Allocations with Thresholds, Preliminary Version of a Discussion Paper, September 20th 2005, Department of Law and Economics Martin-Luther-University Halle-Wittenberg.

Ahlert, Marlies (2006): Discrete Allocation of a Divisible Good – Allocation of Chances, Discussion Papers in Economics, No.50, Department of Law and Economics Martin-Luther-University Halle-Wittenberg.

Ahlert, Marlies (2007): If not only Numbers Count – Allocation of Equal Chances, Preliminary Version of a Discussion Paper, November 25th 2007, Department of Law and Economics Martin-Luther-University Halle-Wittenberg.

Imai, Haruo (1983): Individual Monotonicity and Lexicographic Maximin Solution, in: *Econometrica*, Vol. 51, No. 2, pp. 389-401.

Kalai, Ehud and Meir Smorodinsky (1975): Other Solutions to Nash's Bargaining Problems, *Econometrica*, Vol. 43, pp. 513 – 518.

Lübbe, Weyma (2004): Tödliche Entscheidung: Allokation von Leben und Tod in Zwangslagen, Mentis, Paderborn.

Nash, John F. Jr. (1950): The Bargaining Problem, in *Econometrica*, Vol. 18, No. 2, pp. 155 – 162.

Parfit, Derek (1978): Innumerate Ethics, in: *Philosophy and Public Affairs*, Vol. 7 (4), pp. 285 – 301.

Roth, Alvin E. (1979a): An Impossibility Result Concerning n-Person Bargaining Games, in: *International Journal of Game Theory*, Vol. 8, Issue 3, pp. 129-132.

Roth, Alvin E. (1979b): Axiomatic Models of Bargaining, eds: Beckmann, M; Künzi, H.P., Springer, Berlin.

Taurek, John (1977): Should the Numbers Count?, in: *Philosophy and Public Affairs*, Vol. 6, pp. 293 – 316.

Thomson, William (1994): Cooperative Models of Bargaining, in: *Handbook of Game Theory with Economic Applications*, eds: Aumann, R.; Hart, S., Elsevier Science, Amsterdam, pp. 1237 - 1284.

Zimmermann, Jana (2007): Modellierung von Priorisierungsregeln am Spezialfall der Triage, Preliminary Version of a Discussion Paper, November 28th 2007, Department of Law and Economics Martin-Luther-University Halle-Wittenberg.

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